Computers and Mathematics with Applications 58 (2009) 711–716

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Generalizations of E-convex and B-vex functions[☆]

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ARTICLE INFO

Article history:
Received 16 February 2009
Accepted 14 April 2009

Keywords:
Generalized convexity
B-vex functions
E-convex functions
E-quasiconvex functions
E-B-vex functions

ABSTRACT

A class of functions called E-B-vex functions is defined as a generalization of E-convex and B-vex functions. Similarly, a class of E-B-preinvex functions, which are generalizations of E-convex and B-preinvex functions, is introduced. In addition, the concept of B-linear functions is also generalized to E-B-linear functions. Some properties of these proposed classes are studied. Furthermore, the equivalence between the class of E-B-vex functions and that of E-quasiconvex functions is proved.

1. Introduction

Convexity and generalized convexity play important roles in optimization theory. Various generalizations of convexity have appeared in the literature. A significant generalization of convex functions is preinvex functions, introduced by Hanson and Mond [1] but so named by Jeyakumar [2]. Recently, another generalization of convex functions, called B-vex functions, was introduced by Bector and Singh [3]. Later, Suneja et al. [4] introduced a class of functions called B-preinvex functions which are generalizations of preinvex and B-vex functions. Li et al. [5] proved that the class of B-vex functions is equivalent to that of quasiconvex functions.

Youness [6] introduced a class of sets and a class of functions, called E-convex sets and E-convex functions, which generalize the definitions of convex sets and convex functions based on the effect of an operator E on the sets and domain of definition of the functions. The initial results of Youness [6] inspired a great deal of subsequent work which has greatly expanded the role of E-convexity in optimization theory; see for example [7–11]. In an earlier paper [10], we introduced a class of functions, called E-quasiconvex functions, which are a generalization of E-convex functions and quasiconvex functions. Fulga and Preda [9] extended the classes of preinvex and E-convex functions to E-preinvex functions, and the class of E-quasiconvex functions to E-prequasiinvex functions.

Motivated both by earlier research works [3,9,4,10,6] and by the importance of convexity and generalized convexity, we introduce a class of functions called E-B-vex functions which are generalizations of E-convex and B-vex functions, and a class of E-B-preinvex functions which are generalizations of E-convex and B-preinvex functions. In addition, the concept of B-linear functions is also generalized to E-B-linear functions. Some properties of these proposed classes are studied. Furthermore, the equivalence between the class of E-B-vex functions and that of E-quasiconvex functions is proved.

☆ Supported by the National Science Council of Taiwan under contract NSC 96-2221-E-150-012.

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doi:10.1016/j.camwa.2009.04.012

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space; let X be a nonempty subset of R^n; let R^* denote the set of nonnegative real numbers; let $b : X \times X \times [0, 1] \to R^*$, with $\lambda b(x, y, \lambda) \in [0, 1]$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Following along the lines of Bector and Singh [3], the definitions of B-vex and B-linear functions can be given as follows.

Definition 2.1 (*Ref. [5]*). Let $C \subseteq X$ be a nonempty convex set. A function $f : C \to R^1$ is said to be:

(1) B-vex on C with respect to (w.r.t. in short) $b(x, y, \lambda)$ if for all $x, y \in C$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \le \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y);$$

(2) B-linear on C w.r.t. $b(x, y, \lambda)$ if for all $x, y \in C$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) = \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y).$$

For the sake of brevity, we shall omit the argument of b unless it is needed for specification.

Recall [12] that, by definition, a set $K \subseteq R^n$ is called an invex set w.r.t. a given mapping $\eta : R^n \times R^n \to R^n$ if

$$x, y \in K, \quad \lambda \in [0, 1] \Longrightarrow y + \lambda\eta(x, y) \in K.$$

A set $M \subseteq R^n$ is said to be E-convex if there is a mapping $E : R^n \to R^n$ such that

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall x, y \in M, \ \forall \lambda \in [0, 1].$$

In what follows, let $E : R^n \to R^n$ and $\eta : R^n \times R^n \to R^n$ be two fixed mappings. Now, we state the concept of E-invex sets which are generalizations of invex and E-convex sets.

Definition 2.2 (*Ref. [9, Definition 2.2]*). A set $A \subseteq R^n$ is said to be E-invex w.r.t η if

$$x, y \in A, \quad \lambda \in [0, 1] \Longrightarrow E(y) + \lambda\eta(E(x), E(y)) \in A.$$

Let S be a nonempty subset of R^n; $E(S)$ is defined as follows:

$$E(S) = \{E(x) : x \in S\}.$$

Lemma 2.1 (*Ref. [6, Proposition 2.2]*). *Let $M \subseteq R^n$ be a nonempty E-convex set; then $E(M) \subseteq M$.*

Lemma 2.2 (*Ref. [9, Lemma 2.1]*). *Let $A \subseteq R^n$ be a nonempty E-invex set; then $E(A) \subseteq A$.*

Finally, we describe several generalized convex functions, viz. preinvex, B-preinvex, E-convex, E-quasiconvex, and E-preinvex.

Definition 2.3 (*Ref. [12]*). Let $K \subseteq R^n$ be a nonempty invex set w.r.t. η. A function $f : K \to R^1$ is said to be preinvex on K w.r.t. η, if for all $x, y \in K$ and $\lambda \in [0, 1]$,

$$f(y + \lambda\eta(x, y)) \le \lambda f(x) + (1-\lambda)f(y).$$

Definition 2.4 (*Ref. [4]*). Let $K \subseteq X$ be a nonempty invex set w.r.t. η. A function $f : K \to R^1$ is said to be B-preinvex on K w.r.t. η, b, if for all $x, y \in K$ and $\lambda \in [0, 1]$,

$$f(y + \lambda\eta(x, y)) \le \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y).$$

Definition 2.5 (*Refs. [10,6]*). Let $M \subseteq R^n$ be a nonempty E-convex set. A function $f : M \to R^1$ is said to be:

(1) E-convex on M if for all $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda)f(E(y));$$

(2) E-quasiconvex on M if for all $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \max\{f(E(x)), f(E(y))\};$$

(3) E-quasiconcave on M if for all $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1-\lambda)E(y)) \ge \min\{f(E(x)), f(E(y))\}.$$

Definition 2.6 (*Ref. [9, Definition 2.3]*). Let $A \subseteq R^n$ be a nonempty E-invex set w.r.t. η. A function $f : A \to R^1$ is said to be E-preinvex on A w.r.t. η if for all $x, y \in A$ and $\lambda \in [0, 1]$,

$$f(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f(E(x)) + (1-\lambda)f(E(y)).$$

3. Basic results

First, a class of E-B-vex functions is introduced as a generalization of E-convex and B-vex functions, and the concept of B-linear functions is also generalized to E-B-linear functions.

Definition 3.1. Let $M \subseteq X$ be a nonempty E-convex set. A function $f : M \to R^1$ is said to be:

(1) E-B-vex on M w.r.t. b if $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y));$$

(2) E-B-linear on M w.r.t. b if $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1-\lambda)E(y)) = \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y)).$$

Remark 3.1. Let $M \subseteq R^n$ be a nonempty E-convex set. It follows from Lemma 2.1 that $E(M) \subseteq M$. Hence, for any $f : M \to R^1$, the restriction $\tilde{f} : E(M) \to R^1$ of $f : M \to R^1$ to $E(M)$ defined by

$$\tilde{f}(\tilde{x}) = f(\tilde{x}) \quad \text{for all } \tilde{x} \in E(M)$$

is well defined.

Let $M \subseteq X$ be a nonempty E-convex set. Direct examination of the definition of E-B-vex (resp. E-B-linear) functions shows that the set of E-B-vex (resp. E-B-linear) functions on M w.r.t. the same b is closed under addition and nonnegative scalar multiplication. This is formalized in the following theorem.

Theorem 3.1. *Let $M \subseteq X$ be a nonempty E-convex set, and let $\alpha \geq 0$. If f and g are E-B-vex (resp. E-B-linear) functions on M w.r.t. the same b, then $f + g$ and αf are E-B-vex (resp. E-B-linear) functions on M w.r.t. b.*

Corollary 3.1. *Let $M \subseteq X$ be a nonempty E-convex set. Let f_j, $j = 1, 2, \ldots, N$ be E-B-vex (resp. E-B-linear) functions on M w.r.t. the same b. Then the function $f : M \to R^1$ defined by*

$$f(x) = \sum_{j=1}^{N} k_j f_j(x), \quad k_j \geq 0,$$

is E-B-vex (resp. E-B-linear) on M w.r.t. b.

Next, we introduce a new class of functions called E-B-preinvex functions by relaxing the definitions of E-convex and B-preinvex functions.

Definition 3.2. Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η. A function $f : A \to R^1$ is said to be E-B-preinvex on A w.r.t. η, b, if for all $x, y \in A$ and $\lambda \in [0, 1]$,

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y)).$$

Remark 3.2. Let $A \subseteq R^n$ be a nonempty E-invex set. It follows from Lemma 2.2 that $E(A) \subseteq A$. Hence, for any $f : A \to R^1$, the restriction $\tilde{f} : E(A) \to R^1$ of $f : A \to R^1$ to $E(A)$ defined by

$$\tilde{f}(\tilde{x}) = f(\tilde{x}) \quad \text{for all } \tilde{x} \in E(A)$$

is well defined.

An analogous result to Theorem 3.1 for the E-B-preinvex case is as follows.

Theorem 3.2. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η, and let $\alpha \geq 0$. If f and g are E-B-preinvex functions on A w.r.t. the same η, b, then $f + g$ and αf are E-B-preinvex functions on A w.r.t. η, b.*

Corollary 3.2. *Let $A \subseteq X$ be a nonempty E-invex set. Let f_j, $j = 1, 2, \ldots, N$ be E-B-preinvex functions on A w.r.t. the same η, b. Then the function $f : A \to R^1$ defined by*

$$f(x) = \sum_{j=1}^{N} k_j f_j(x), \quad k_j \geq 0,$$

is E-B-preinvex on A w.r.t. η, b.

Finally, we derive a property of E-preinvex functions.

Theorem 3.3. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η. Suppose that $f : A \to R^1$ is E-preinvex on A w.r.t. η, and that $\phi : R^1 \to R^1$ is nondecreasing and convex. Then $\phi \circ f : A \to R^1$ is E-preinvex on A w.r.t. η.*

Proof. Since $f : A \to R^1$ is E-preinvex on A w.r.t. η, and $\phi : R^1 \to R^1$ is nondecreasing and convex, we have, for any $x, y \in A$ and $\lambda \in [0, 1]$,

$$\begin{aligned}\phi \circ f(E(y) + \lambda\eta(E(x), E(y))) &= \phi(f(E(y) + \lambda\eta(E(x), E(y))))\\ &\leq \phi(\lambda f(E(x)) + (1-\lambda)f(E(y)))\\ &\leq \lambda\phi(f(E(x))) + (1-\lambda)\phi(f(E(y)))\\ &= \lambda\phi \circ f(E(x)) + (1-\lambda)\phi \circ f(E(y)).\end{aligned}$$

That is, $\phi \circ f : A \to R^1$ is E-preinvex on A w.r.t. η. □

4. Main results

We first study the relations between E-B-vex and B-vex (resp. E-quasiconvex) functions.

Theorem 4.1. *Let $M \subseteq X$ be a nonempty E-convex set, and let C be a nonempty convex subset of $E(M)$. If $f : M \to R^1$ is E-B-vex on M w.r.t. b, then the restriction $\hat{f} : C \to R^1$ of $f : M \to R^1$ to C defined by*

$$\hat{f}(\hat{x}) = f(\hat{x}) \quad \textit{for all } \hat{x} \in C$$

is a B-vex function on C w.r.t. b.

Proof. Let $f : M \to R^n$ be E-B-vex on M w.r.t. b, and let C be a nonempty convex subset of $E(M)$. Then for $\hat{x}, \hat{y} \in C$ ($\hat{x}$ and $\hat{y}$ may not be distinct), there exist $x, y \in M$ such that $\hat{x} = E(x)$ and $\hat{y} = E(y)$. Since $\lambda\hat{x} + (1-\lambda)\hat{y} \in C$, it follows from the E-B-vexity of f on M that

$$\begin{aligned}\hat{f}(\lambda\hat{x} + (1-\lambda)\hat{y}) &= f(\lambda E(x) + (1-\lambda)E(y))\\ &\leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y))\\ &= \lambda b(\hat{x}, \hat{y}, \lambda)\hat{f}(\hat{x}) + (1 - \lambda b(\hat{x}, \hat{y}, \lambda))\hat{f}(\hat{y})\end{aligned}$$

for all $\lambda \in [0, 1]$, which implies that $\hat{f} : C \to R^1$ is a B-vex function on C w.r.t. b. □

Corollary 4.1. *Let $M \subseteq X$ be a nonempty E-convex set, and let $f : M \to R^1$ be E-B-vex on M w.r.t. b. If $E(M)$ is a convex set, then the restriction $\tilde{f} : E(M) \to R^1$ of $f : M \to R^1$ is a B-vex function on $E(M)$ w.r.t. b.*

Theorem 4.2. *Let $M \subseteq X$ be a nonempty E-convex set such that $E(M)$ is convex. Then a function $f : M \to R^1$ is E-B-vex on M w.r.t. b if and only if its restriction $\tilde{f} : E(M) \to R^1$ is B-vex on $E(M)$ w.r.t. b.*

Proof. The direct implication is true due to Corollary 4.1. Conversely, suppose that $\tilde{f} : E(M) \to R^1$ is a B-vex function on $E(M)$ w.r.t. b, and that $x, y \in M$. Then $E(x), E(y) \in E(M)$, and by the convexity of $E(M)$ follows $\lambda E(x) + (1-\lambda)E(y) \in E(M)$ for all $\lambda \in [0, 1]$. Since $\tilde{f} : E(M) \to R^1$ is B-vex on $E(M)$ w.r.t. b, we have

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda b(E(x), E(y), \lambda)f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda))f(E(y))$$

for all $\lambda \in [0, 1]$, which implies that $f : M \to R^1$ is E-B-vex on M w.r.t. b. This completes the proof. □

The following result can be easily established.

Theorem 4.3. *Let $M \subseteq X$ be a nonempty E-convex set, and let $\{f_j : j \in J\}$ be an arbitrary nonempty collection of E-B-vex functions on M w.r.t. the same b such that for each $x \in M$, $\sup_{j\in J} f_j(x)$ exists in R^1. Then the function $f : M \to R^1$ defined by*

$$f(x) = \sup_{j\in J} f_j(x) \quad \textit{for each } x \in M,$$

is E-B-vex on M.

The following theorem which can be established along the lines of Theorem 2.1 of Li et al. [5] presents the equivalence between the class of E-B-vex functions and that of E-quasiconvex functions. For the convenience of reading, the proof will be given.

Theorem 4.4. *Let $M \subseteq X$ be a nonempty E-convex set. The following conditions are equivalent:*

(1) *$f : M \to R^1$ is E-B-vex on M w.r.t. some b.*
(2) *$f : M \to R^1$ is E-quasiconvex on M.*

Proof. (1) $\Longrightarrow$ (2) Let f be E-B-vex on M w.r.t. b. Noting that $\lambda b(E(x), E(y), \lambda) \in [0, 1]$ for all $x, y \in M$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &\le \lambda b(E(x), E(y), \lambda) f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda)) f(E(y)) \\ &\le \max\ \{f(E(x)), f(E(y))\}, \end{aligned}$$

for all $x, y \in M$ and $\lambda \in [0, 1]$. This shows that $f : M \to R^1$ is E-quasiconvex on M.

(2) $\Longrightarrow$ (1) Define $b : X \times X \times [0, 1] \to R^*$ by

$$b(x, y, \lambda) = \begin{cases} 1/\lambda, & \text{if } \lambda \in (0, 1] \text{ and } f(x) \ge f(y); \\ 0, & \text{if } \lambda = 0 \text{ or } f(x) < f(y). \end{cases}$$

It follows that $\lambda b(x, y, \lambda) \in [0, 1]$ for all $x, y \in X$ and $\lambda \in [0, 1]$, and that

$$\lambda b(E(x), E(y), \lambda) f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda)) f(E(y)) = \max\ \{f(E(x)), f(E(y))\}$$

for all $x, y \in X$ and $\lambda \in (0, 1]$. Then, by E-quasiconvexity of f on M, we have

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &\le \max\ \{f(E(x)), f(E(y))\} \\ &= \lambda b(E(x), E(y), \lambda) f(E(x)) + (1 - \lambda b(E(x), E(y), \lambda)) f(E(y)), \end{aligned}$$

for all $x, y \in X$ and $\lambda \in (0, 1]$, which implies that $f : M \to R^1$ is E-quasiconvex on M. This completes the proof. □

Using a similar argument to that used in establishing the equivalence of E-B-vexity and E-quasiconvexity of functions, the following result can be easily proved.

Theorem 4.5. *Let $M \subseteq X$ be a nonempty E-convex set. The following conditions are equivalent:*

(1) *$f : M \to R^1$ is E-B-linear on M w.r.t. some b.*
(2) *$f : M \to R^1$ is both E-quasiconvex and E-quasiconcave on M.*

The following theorem which is an analogous result to Theorem 4.1 for the E-B-preinvex case can be easily established along the lines of Theorem 4.1.

Theorem 4.6. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η, and let K be a nonempty invex subset of $E(M)$ w.r.t. η. If $f : A \to R^1$ is E-B-preinvex on A w.r.t. η, b, then the restriction $\hat{f} : K \to R^1$ of $f : A \to R^1$ to K defined by*

$$\hat{f}(\hat{x}) = f(\hat{x}) \quad \text{for all } \hat{x} \in K$$

is a B-preinvex function on K w.r.t. b.

Corollary 4.2. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η, and let $f : A \to R^1$ be E-B-preinvex on A w.r.t. η, b. If $E(A)$ is an invex set w.r.t. η, then the restriction $\tilde{f} : E(A) \to R^1$ of $f : A \to R^1$ is a B-preinvex function on $E(A)$ w.r.t. b.*

The following theorem which is an analogous result to Theorem 4.2 for the E-B-preinvex case can be easily established along the lines of Theorem 4.1.

Theorem 4.7. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η such that $E(A)$ is invex w.r.t. η. Then a function $f : A \to R^1$ is E-B-preinvex on A w.r.t. η, b if and only if its restriction $\tilde{f} : E(A) \to R^1$ is a B-preinvex function on $E(A)$ w.r.t. b.*

The following result can be easily established.

Theorem 4.8. *Let $A \subseteq X$ be a nonempty E-invex set w.r.t. η. If $\{f_j : j \in J\}$ is an arbitrary nonempty collection of E-B-preinvex functions on A w.r.t. the same η, b such that for each $x \in A$, $\sup_{j \in J} f_j(x)$ exists in R^1, then the function $f : A \to R^1$ defined by*

$$f(x) = \sup_{j \in J} f_j(x) \quad \text{for each } x \in A,$$

is E-B-preinvex on A w.r.t. η, b.

References

[1] M.A. Hanson, B. Mond, Convex transformable programming problems and invexity, Journal of Information and Optimization Sciences 8 (1987) 201–207.
[2] W. Jeyakumar, Strong and weak invexity in mathematical programming, European Journal of Operational Research 55 (1985) 109–125.
[3] C.R. Bector, C. Singh, B-vex functions, Journal of Optimization Theory and Applications 71 (1991) 237–254.

[4] S.K. Suneja, C. Singh, C.R. Bector, Generalization of preinvex and B-vex functions, Journal of Optimization Theory and Applications 76 (1993) 577–587.
[5] X.F. Li, J.L. Dong, Q.H. Liu, Lipschitz B-vex functions and nonsmooth programming, Journal of Optimization Theory and Applications 3 (1997) 557–573.
[6] E.A. Youness, E-convex sets, E-convex functions and E-convex programming, Journal of Optimization Theory and Applications 102 (1999) 439–450.
[7] D.I. Duca, E. Duca, L. Lupsa, R. Blaga, E-convex functions, Bulletin for Applied and Computer Mathematics 43 (2000) 93–102.
[8] D.I. Duca, L Lupsa, On the E-epigraph of an E-convex Function, Journal of Optimization Theory and Applications 120 (2006) 341–348.
[9] C. Fulga, V. Preda, Nonlinear programming with E-preinvex and Local E-preinvex functions, European Journal of Operational Research 192 (2009) 737–743.
[10] Y.R. Syau, E.S. Lee, Some properties of E-convex functions, Applied Mathematics Letters 18 (2005) 1074–1080.
[11] X.M. Yang, E-convex sets, E-convex functions and E-convex programming, Journal of Optimization Theory and Applications 109 (2001) 699–704.
[12] S.R. Mohan, S.K. Neogy, On invex sets and preinvex functions, Journal of Mathematical Analysis and Applications 189 (1995) 901–908.